Coordinate Vectors

Last Class

We saw that every vector space has a basis
- turns out that the # of vectors in a basis for a given VS is fixed
  - Call this # the dimension of V

ex) $\dim(\mathbb{R}^n) = n$

$\dim(\mathbb{R}_n[x]) = n+1$

$\dim(M_{m \times n}(\mathbb{R})) = mn$

Section 5.4 – Coordinate vectors / Change of Basis

Recall: In checking if the vector

i) $(2+3x-5x^2)$ in span$(1+x-2x^2, 2+x-3x^2)$

$$\Updownarrow$$

Checking if the 3-vector $\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$ in span$\left(\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}\right)$

ii) Checking if $\begin{pmatrix} 4 & 1 \\ -2 & -3 \end{pmatrix}$ in span$\left(\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}\right)$

$$\Updownarrow$$

Checking if 4-vector $\begin{pmatrix} 4 \\ 1 \\ -2 \\ -3 \end{pmatrix}$ in span $\left(\begin{pmatrix} 2 \\ 1 \\ 9 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} \right)$

Why care about Basis in first place?

Let $\mathcal{B} = (v_1, \dots, v_n)$ be a basis for V

- then every vector w in V can be expressed uniquely as the sum

$$w = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

Def: The coordinate vector for w with respect to a basis $\mathcal{B}$

$$[w]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \text{ in } \mathbb{R}^n$$

ex) $V = \mathbb{R}_2[x]$, $\mathcal{B} = (1, x, x^2)$

- then $3 - x + 5x^2 = 3(1) - 1(x) + 5(x^2)$

so $[3 - x + 5x^2]_{\mathcal{B}} = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$

- $\mathcal{B}' = (2+x, 3+x^2, x - x^2)$ is another basis for V

- then $3 - x + 5x^2 = c_1(2+x) + c_2(3+x^2) + c_3(x - x^2)$

$3 - x + 5x^2 = 9(2+x) - 5(3+x^2) - 5(x - x^2)$ ← turns out

$\Rightarrow [3 - x + 5x^2]_{\mathcal{B}'} = \begin{pmatrix} 9 \\ -5 \\ -5 \end{pmatrix}$

Thrm: Let V vs and $\mathcal{B} = (v_1 \dots v_n)$ a basis.

Then a vector w in span$(u_1, u_2, \dots, u_m)$ if and only if

$$[w]_{\mathcal{B}} \text{ in span} \left([u_1]_{\mathcal{B}}, [u_2]_{\mathcal{B}}, \dots, [u_m]_{\mathcal{B}} \right)$$

We already saw this!

ex) $V = M_{2\times 2}(\mathbb{R})$, $\mathcal{B} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$

Then $\begin{pmatrix} 4 & 1 \\ -2 & -3 \end{pmatrix}$ in span$\left(\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \right)$ if and only if

$$\left[\begin{pmatrix}4 & 1\\ -2 & -3\end{pmatrix}\right]_{\mathcal{B}} \text{ in span}\left(\left[\begin{pmatrix}2 & 1\\ 0 & 0\end{pmatrix}\right]_{\mathcal{B}}, \left[\begin{pmatrix}0 & 1\\ 2 & 3\end{pmatrix}\right]_{\mathcal{B}}\right)$$

$$\Vert$$

$$\begin{pmatrix}4\\ 1\\ -2\\ -3\end{pmatrix} \text{ in span}\left(\begin{pmatrix}2\\ 1\\ 0\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 1\\ 2\\ 3\end{pmatrix}\right) ?$$

ii) $V = \mathbb{R}_2[x]$, $\mathcal{B} = (1, x, x^2)$

Checking if $2+3x-5x^2$ in span$(1+x-2x^2, 2+x-3x^2)$?

$$\Vert$$

$$[2+3x-5x^2]_{\mathcal{B}} \text{ in span}\left([1+x-2x^2]_{\mathcal{B}}, [2+x-3x^2]_{\mathcal{B}}\right) ?$$

$$\Vert$$

Check if $\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$ in span $\left(\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \right)$?

What this thm tells us is questions about spanning in general can be reduced to questions about spanning in $\mathbb{R}^n$

$\Downarrow$

<u>Thm</u>: V n-dim VS. Let $(w_1, \dots, w_k)$ be vectors in V

1) If $(w_1, \dots, w_k)$ span V then $k \geq n$
2) If $(w_1, \dots, w_k)$ is LI then $k \leq n$

Also have "half is good enough" statements (see HW) ex) $V = \mathbb{R}_2[x]$ $\quad (1, x, 2x, 3x)$ - have 4 vectors in 3-dim space

but do not span.

ex) $V = M_{2\times 2}(\mathbb{R})$ $\quad \left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right)$

- have 3 vectors in 4-dim space, but they are not LI.

Recall: We saw that $[w]_{\mathcal{B}}$ changes depending on our basis.

$V = \mathbb{R}_2[x]$

- $\mathcal{B} = (1, x, x^2)$
- $\mathcal{B}' = (2+x, 3+x^2, x-x^2)$

$$\Rightarrow \quad [3-x+5x^2]_{\mathcal{B}} = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$$

$$[3-x+5x^2]_{\mathcal{B}'} = \begin{pmatrix} 9 \\ -5 \\ -5 \end{pmatrix}$$

Are there any relationship between $[w]_{\mathcal{B}}$ and $[w]_{\mathcal{B}'}$

Yes! - The relationship is given as follows:

Def: Let V n-dim vector space. Let

$\mathcal{B} = (v_1, \dots, v_n)$ and $\mathcal{B}' = (w_1, \dots, w_n)$ be 2 basis.

The matrix

$$P_{\mathcal{B}\to\mathcal{B}'} = \begin{pmatrix} [v_1]_{\mathcal{B}'} & [v_2]_{\mathcal{B}'} & \cdots & [v_n]_{\mathcal{B}'} \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}_{n\times n}$$

is called the change of basis matrix from $\mathcal{B}\to\mathcal{B}'$

Thm: V n-dim VS, $\mathcal{B}, \mathcal{B}'$ 2 diff basis.

Then $[w]_{\mathcal{B}'} = P_{\mathcal{B}\to\mathcal{B}'}[w]_{\mathcal{B}}$

- Moreover $\left(P_{\mathcal{B}\to\mathcal{B}'}\right)^{-1} = P_{\mathcal{B}'\to\mathcal{B}}$

ex) $V = \mathbb{R}_2[x]$ and $\mathcal{B} = (1, x, x^2)$

$\mathcal{B}' = (1, 2x + 4x^2, x^2)$

Let $f = 2 - 6x + 3x^2$

a) Find $[f]_{\mathcal{B}}$

- $f = 2(1) - 6(x) + 3(x^2)$

$\Rightarrow [f]_{\mathcal{B}} = \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix}$

b) Find $P_{\mathcal{B} \to \mathcal{B}'}$

$1 = 1(1) + 0(2x+4x^2) + 0(x^2) \Rightarrow [1]_{\mathcal{B}'} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$x = 0(1) + \frac{1}{2}(2x+4x^2) - 2(x^2) \Rightarrow [x]_{\mathcal{B}'} = \begin{pmatrix} 0 \\ 1/2 \\ -2 \end{pmatrix}$

$$x^2 = 0(1) + 0(2x + 4x^2) + 1(x^2) \Rightarrow [x^2]_{B'} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$P_{B \to B'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

c) Find $[f]_{B'}$

Thrm tells us $[f]_{B'} = P_{B \to B'} [f]_B$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix}$$

$$= 2\begin{pmatrix}1\\0\\0\end{pmatrix} - 6\begin{pmatrix}0\\1/2\\-2\end{pmatrix} + 3\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix}2\\0\\0\end{pmatrix} + \begin{pmatrix}0\\-3\\12\end{pmatrix} + \begin{pmatrix}0\\0\\3\end{pmatrix} = \begin{pmatrix}2\\-3\\15\end{pmatrix}$$

Check: Is $2(1) - 3(2x+4x^2) + 15(x^2) = f$?

$$2 - 6x - 12x^2 + 15x^2 = 2 - 6x + 3x^2 = f \quad \checkmark$$

d) Check that $(P_{\mathcal{B}\to\mathcal{B}'})^{-1} = P_{\mathcal{B}'\to\mathcal{B}}$

$$1 = 1(1) + 0(x) + 0(x^2) \implies [1]_{\mathcal{B}} = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$2x+4x^2 = 0(1) + 2(x) + 4(x^2) \implies [2x+4x^2]_{\mathcal{B}} = \begin{pmatrix}0\\2\\4\end{pmatrix}$$

$$x^2 = 0(1) + 0(x) + 1(x^2) \implies [x^2]_{\mathcal{B}} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$P_{B' \to B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 4 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$